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AN ACCELERATED METHOD FOR CALCULATING THE EXNER FUNCTION

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An Accelerated Method for Calculating the Exner Function

Tech Note

In the sigma coordinate system, it is necessary to evaluate the expression $\Pi = (P/1000)^{R/CP}$ at each point each time step. In a FORTRAN program on the CDC 6600, this was found to be very expensive. The compiler evaluates the logarithm, performs the multiplication, then evaluates the exponential. Earlier at the NMC, this was improved by means of a polynomial expression. This was a noticeable improvement but required a rather large number of terms to get a good fit at high levels. Recently some efforts have been made to improve the logarithmic routine at Suitland. This paper extends the work to the expression

$$Y = X{*}{*}B \qquad (1)$$

A routine to carry this out has been produced for the CDC 6600 that will run as a stack resident loop and evaluate the expression at the rate of about 10μ per point and produce, at the worst, 8-place accuracy over the full meteorological range of the variable. The following development explains the theory:

Given:

$$y = x^b \qquad (2)$$

This can be expanded in Taylor's series.

$$y = a^b + (x-a)ba^{(b-1)} + \frac{(x-a)^2}{2!}b(b-1)a^{b-2} + \ldots \qquad (3)$$

where a is some value nearby x whose solution a^b is known a^b can be factored out from (3) giving

$$y = a^b\left(1 + \frac{(x-a)b}{a} + \frac{(x-a)^2}{a^2}\,\frac{b(b-1)}{2!} + \ldots\right. \qquad (4)$$

The series converges if $-1 < \frac{x-a}{a} < 1$

The error term after n terms is approx $\frac{1}{(nh)}\left(\frac{x-a}{a}\right)^{nh}$

This means that $\frac{x-a}{a}$ must be very small (close guess) or else the series coverages very slowly.

The CDC 6600 has two multiplier units so another useful form is

$$y = a^b \frac{1+X(K_1 + X(K_2 + \ldots}{1+X(L_1 + X(L_2 + \ldots} \qquad (5)$$

This has the advantage that the numerator and denominator can be evaluated in parallel.

To evaluate the K_i L_i long division is performed and coefficients of equal powers of X in (4) and (5) are equated.

For example if i is restricted to 2 then

$$y = a^b \frac{1+X(K_1 + K_2X)}{1+X(L_1 + L_2X)}$$

If the series terms are labeled

$$\text{Term } 1 = b$$

$$\text{Term } 2 = \frac{b(b-1)}{2!}$$

etc.

Carrying out division and substituting gives

$$L_2 = -\text{Term } 4/\text{Term } 2$$

$$L_1 = -(\text{Term } 3 + \text{Term } 1(L_2)\ )\ /\ \text{Term } 2$$

$$K_2 = -\text{Term } 2 + L_2 + L_1/\text{Term } 1$$

$$K_1 = -\text{Term } 1 + L_1$$

Now in order to use such a short series, it is necessary to examine a good way to get a close value of a^b. First of all we note that any factors of 2 can be taken out easily since

$$(2^n\ a)^b = (2)^b\ (a^b) = (2^b)^n\ (a^b).$$

This allows us to take advantage of the fact that a is presented to us as a binary floated number. Its exponent can be separated by the unpack instruction and used to look up $(2^{R/CP})^n$. A table ranging from 0 to N = -9 will cover the full meteorological range of the variable. Then the first M bits of the fractional part can be used as a look-up argument so that if M is 4 $\frac{x-a}{a}$ will be less than 1/16 for the whole range of the variable. This will make the approximation be good to about 8 places in the meteorological range.

A copy of the pertinent parts of the code is attached. The FORTRAN function evaluates the expressions L_2, L_1, K_2, K_1, eight values of $(2\ \mathrm{NOCP})^N$, and eight values of $(M/16)^{R/CP}$. The machine language loop assumes p/1000 is already in the array PI and that register B3 contains 1.

Appendix

Attached are some sheets for the assembly listing of the operational code. The FORTRAN statements are from STARTF which computes the required constants once and for all in the initialization. The constant field Picon has two parts. 1-8 are the powers of 2**ROCP. The second part contains 8 values each of L_2/a_i, L_1/a_i, K_2/a_i, K_1/a in that order respectively. Even though there are potentially 16 values in four bits, the requirement that the fraction be normalized means that 0 through 7 do not occur. Page 2 and 3 show an extract from the tendency calculation where π is calculated for all I and K on a single J strip.

```
      C COMPUTE THE CONSTANTS FOR FAST A**B
            XX=ROCP
            TERM(1)=XX
            DO 100 I=2,4
            XX = XX-1.
            YY = I
        100 TERM(I)=TERM(I-1)*XX/YY
            PICON(2)=(TERM(3)**2-TERM(2)*TERM(4))/(TERM(2)**2-TERM(1)*TERM(3))
            PICON(4)=-(TERM(3)+TERM(1)*PICON(2))/TERM(2)
            PICON(1)=TERM(2)+PICON(2)+PICON(4)*TERM(1)
            PICON(3)=TERM(1)+PICON(4)
            DO 110 I=1,8
            XX=I+7
            YY=XX/16.
            ZZ=YY**2
            PICON(I+8)=PICON(1)/ZZ
            PICON(I+16)=PICON(2)/ZZ
            PICON(I+24)=PICON(3)/YY
            PICON(I+32)=PICON(4)/YY
        110 PICON(I+40)=YY**ROCP
      C POWERS OF 2**RCCP
            PICON(8)=2.**ROCP
            DO 120 I=2,8
            J=9-I
        120 PICON(J)=PICON(J+1)/PICON(8)
```

```
                                    * SOLVE  FOR  PI AND PHI
                                    * THE CONSTANTS ARE PRECOMPUTED IN STARTF AND SAVED IN PICON
                                    * THIS ROUTINE SHOULD BE A VAST IMPROVEMENT IN SPEED
                                    * IT EVALUATES  PI= (PSIG/1000)**ROCP
                                    * IT SHOULD HAVE WORST CASE 8 PLACE ACCURACY PVER THE ENTIRE RANGE
                                    * PRESSURE RANGE IS ZMB TO 2060MB ,VARIABLE IS NOT CHECKED FOR INRANG
                                    *THIS IS A SPECIAL ROUTINE TO CALCULATE THE WHOLE PI STRIP
000205  5110012446 +                         SA1     JN+1
                  5120000035 C            SA2   PICON+6
000206  6120000054                        SB2   44
                  6150000032 C            SB5   PI3+26
000207  10022                             BX0   X2                    FLOATING 1,
             63515                        SB5   B5+X1
                  66451                   SB4   B5+B1
                       67751              SB7   B5-B1
000210  56140                             SA1   B4                    ARGUMENT
             6150777717                      SB5     -48
000211  6110000010                           SB1     8
                  5107000476                 SA0     B7+318         DO ALL LEVELS PRONTO
000212  6170000027 C                         SB7     PICON
                                    *  START ON FIRST POINT
                  26661                      UX6     B6,X1          SEPARATE BINARY MAGNITUDE
                       23726                 AX7     X6,B2          SEPARATE GUESS VALUE
000213  53477                                SA4     B7+X7          K2/A**2
             22727                           LX7     X7,B2
                  54541                      SA5     A4+B1          L2/A**2
                       37767                 IX7     X6-X7          X-A
000214  54251                                SA2     A5+B1          K1/A
             54321                           SA3     A2+B1          L1/A
                                    * MAIN LOOP
000215  27657                         AB4    PX6     B5,X7          FLOAT(X-A)
             54131                           SA1     A3+B1          APPROX. ANSWER
                  6166000066                 SB6     B6+54          REVERSE SCALE
000216  24606                                NX6     X6
             40756                           FX7     X5*X6          START DENOM.
                  56567                      SA5     B6+B7          MAGNITUTE OF ANSWER
                       40446                 FX4     X4*X6          START NUM.
000217  30773                                FX7     X7+X3          DENOM.
```

```
                56343                       SA3    B4+B3     NEXT ARG.
                     30442                  FX4    X4+X2     NUM.
                          40776             FX7    X7*X6     DENOM.
 000220  40446                              FX4    X4*X6     NUM.
                30770                       FX7    X7+X0
                     30440                  FX4    X4+X0
                          40251             FX2    X5*X1     APPROX. ANSWER
 000221  44547                              FX5    X4/X7     CORRECTION
                26663                       UX6    B6,X3     NEXT ARG.
                     23726                  AX7    X6,B2
                          53477             SA4    B7+X7     K2/A**2
 000222  22727                              LX7    X7,B2
                37767                       IX7    X6-X7
                     40625                  FX6    X2*X5     FINAL ANSWER
                          54541             SA5    A4+B1     L2/A**2
 000223  54251                              SA2    A5+B1     K1/A
                54321                       SA3    A2+B1     L1/A
                     66443                  SB4    B4+B3     COUNT
                          75104             SX1    A0-B4     END TEST
 000224  57643                              SA6    B4-B3     STORE RESULTS
                0321000215 +                PL     X1,AB4
```